4/10/23

MATH2050A Tutorial

Announcements:

- HW2 due 10/10
- Midterm1 is on 11/10 (start at 8:30am – 10:00am), Tutorial will be on Friday 13/10 start at 9:30am in LHC G04.

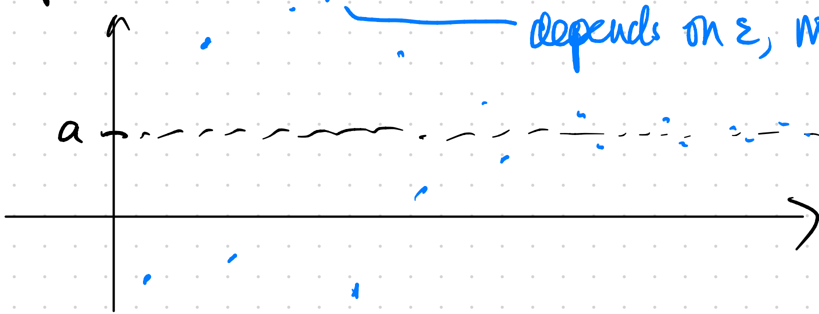
Recap Limits

Def: let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ be a sequence. We say $\{a_n\}$ converges to $a \in \mathbb{R}$

(Write: $a_n \to a$ as $n \to \infty$, $a_n \xrightarrow{n\to\infty} a$, $\lim_{n\to\infty} a_n = a$)

if $\forall \varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ s.t. for all $n \geq N$, $|a_n - a| < \varepsilon$.

depends on ε, not unique.



Fact: $\lim_{n\to\infty} \frac{1}{n} = 0$. (AP: can always find $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$).

General Strategy for proving a limit:

1) let $\varepsilon > 0$ arbitrary.
2) Guess / Have a candidate for what the limit is, a.
3) Find a good estimate $|a_n - a|$.
   - using ordinary inequalities, triangle inequality, ...
4) Using (3), determine $N(\varepsilon)$.
5) Verify that $N(\varepsilon)$ works.

Q1: Use ε-def'n of limit to show

a) $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

b) $\lim_{n\to\infty} \frac{n^2-1}{2n^2+3} = \frac{1}{2}$.

Pf: (a): let $\varepsilon > 0$ be given.

$\left|\frac{n}{n^2+1}\right| < \left|\frac{n}{n^2}\right| = \frac{1}{n}$. By AP., $\exists N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$.

Then for all $n \geq N$

we have $\left|\frac{n}{n^2+1}\right| < \left|\frac{1}{n}\right| < \left|\frac{1}{1/\varepsilon}\right| = \varepsilon$. as required.

(b): let $\varepsilon > 0$ be given.

$$\left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| = \left|\frac{2n^2-2}{4n^2+6} - \frac{2n^2+3}{4n^2+6}\right| = \left|\frac{-5}{4n^2+6}\right| = \frac{5}{4n^2+6} < \frac{5}{4n^2}$$

So for $N > \sqrt{\frac{5}{4\varepsilon}}$. Then for all $n \geq N$, we have

$$\left| \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right| < \frac{5}{4n^2} < \frac{5}{4\left(\sqrt{\frac{5}{4\varepsilon}}\right)^2} = \frac{5}{4} \cdot \frac{4\varepsilon}{5} = \varepsilon.$$

$\frac{5}{4n^2} < \frac{5}{4n}$, then take $N > \frac{5}{4\varepsilon}$. ← N need not be unique.

for $n>1$

Q2: Prove that if $\lim_{n\to\infty} x_n = x$ and $x>0$, then $\exists\, m\in\mathbb{N}$ s.t. $\forall n\geq m$,

$x_n > 0$.

Pf: $\lim_{n\to\infty} x_n = x$ means $\forall \varepsilon>0$, $\exists N(\varepsilon)\in\mathbb{N}$ s.t. $\forall n\geq N$ with $|x_n - x| < \varepsilon$.

So take $\varepsilon = \frac{x}{2}$.

Then $\exists N$ s.t. $\forall n\geq N$,

$$|x_n - x| < \frac{x}{2} \Rightarrow \quad x_n - x < \frac{x}{2}$$

$$x - x_n < \frac{x}{2}. \Rightarrow x_n > \frac{x}{2} > 0.$$

//

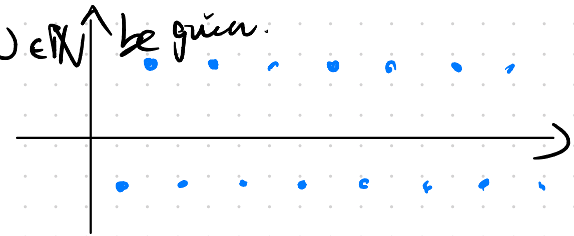
I took $\varepsilon = x+1$

Q3: Prove $\{(-1)^n\}_{n=1}^{\infty}$ does not converge.

"x_n does not converge to x" if $\exists \varepsilon_0 > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t. x_n satisfies

$$|x_n - x| \geq \varepsilon_0.$$

Pf: let $x \in \mathbb{R}$ be given. $\varepsilon_0 = 1$. Let $N \in \mathbb{N}$ be given.



If $x \leq 0$, then take $n = 2k$ for some $k \in \mathbb{N}$ s.t. $k \geq \frac{N}{2}$.

Then $|(-1)^{2k} - x| = |1 - x| \geq 1 = \varepsilon_0$.

since $x \leq 0$.

If $x > 0$, then take $n = 2k+1$ for some $k > \frac{N}{2}$.

Then $|(-1)^{2k+1} - x| = |-1 - x| = 1 + x > 1 = \varepsilon_0$.

Idea: sequence has two subsequences with distinct limits

"Limit exists $\Rightarrow$ all subsequences converge to the same limit."

$\Leftrightarrow$ "two subsequence don't converge to the same value $\Rightarrow$ limit does not exist"

take contrapositive